Dynamic Analysis of a Flexible Vehicle Moving along a Flexible Support

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This paper presents the procedures and rationale of a numeric method for determining the dynamic response of a flexible vehicle moving along a flexible supporting structure. A discrete-element model of the structure is used in the approach which permits a concise matrix formulation of the problem based more on physical than mathematical considerations. The primary factor in this discrete-element analysis is a time-dependent stiffness matrix that is established by a coordinate transformation operation and affected by 1) vehicle movement along the supporting structure and 2) intermittent detachment of the moving vehicle. Any initial curvature of the supporting structure is also taken into account. The development and results of a numerical experiment predicted on this approach are delineated.

I. Introduction

THE general problem involving calculation of the dynamic response of a moving vehicle supported by an elastic structure has been with us for a long time. In fact, over 100 years ago, R. Willis analytically and experimentally investigated the dynamics of a mass moving along, and supported by an elastic structure.[1] His work, an impressive contribution at that time, is still a valuable reference.

Recent approaches to problems of this nature have been based on partial differential equations and mathematical solutions as opposed to numeric solutions. Typical examples of the mathematical approach are Refs. 2–5. Simplifying assumptions were made such as planar motion, a rigid vehicle, and constant contact of the moving vehicle with the supporting structure. These limitations are overcome in the present paper, and intermittent contact of the moving vehicle with the supporting structure due to design geometry, or clearances, is taken into account along with initial curvature of the supporting structure. Needless to say, the mathematical approach is useful for the simpler problem for which it is valid and will continue to find application.

A numeric approach to the calculation of the dynamic response of a combined moving vehicle and supporting structure is presented herein. Based on the use of a discrete-element model of the structure, this approach permits incorporation of elastic properties in a stiffness matrix, which is established by a coordinate transformation and which is affected by 1) vehicle movement along the supporting structure, and 2) intermittent detachment of the moving vehicle from the supporting structure. This time-dependent stiffness matrix is automatically computed during the dynamic analysis by methods which are discussed in later sections of the paper. This specific dynamic response problem is formulated with differential equations which can be solved by a standard step-by-step procedure, such as rail guided, high-speed ground transportation systems.

To illustrate the approach, a specific example is considered, namely, the dynamic response of a missile rail launching. It should be realized that the method applies to a wider range of problems.

For this example, rocket engine thrust propels the missile along the rails, which, in combination with shoes or lugs, provide the missile with initial guidance. The structure considered is shown in Fig. 1. Displacements, velocities, accelerations, and internal forces can be determined for finite intervals as the missile traverses the rails.

II. Typical Discrete-Element Missile Rail Model

A typical discrete-element model with the missile detached from the rails is shown in Fig. 2. The missile is idealized with torsion and beam bending elements. Bending capability is in two planes, the xy and xz planes, respectively. Missile supports which attach the shoes to the missile were added to the model. Their bending and tension flexibilities were adjusted for simulation of the actual missile configuration. The rails in the model have bending and torsion capability. Structure connecting the two rails are idealized as bar and beam elements. The terminology is that bar, beam, and torsion elements react tension (or compression), bending and torsion forces, respectively. Shoe friction forces are disregarded.

The missile shoes do not contact the rails in the principal plane of bending (Fig. 3). This is accounted for by the idealization of stiff vertical bar and beam elements which run from the center of twist of each rail to the points of attachment with the missile shoes.

In short, the missile and rails are idealized as accurately as bar and beam elements permit. A more accurate shell

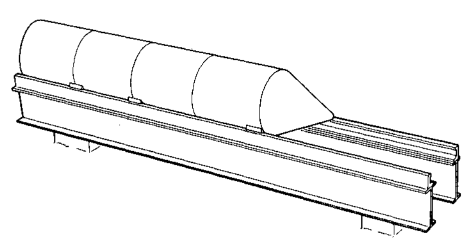

Fig. 1 Missile/rail configuration.

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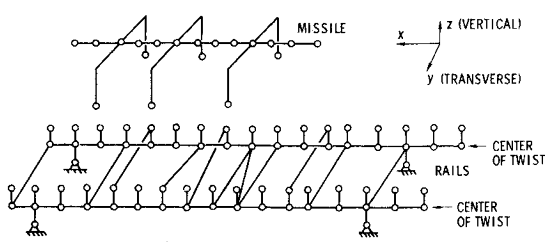


Fig. 2 Discrete-element missile/rail model.

idealization could be used, but the size of the problem and, therefore, the computer run time, would increase.

III. Method of Analysis

A stiffness matrix K_g, which relates all forces to their corresponding coordinates, can be generated for the unassembled structure of Fig. 2. However, in general, forces are not applied for every degree of freedom. The K_g matrix, then, is reduced to the K_z matrix, which is associated with the $\{z\}$ coordinates, which, in turn, correspond to the missile and rail mass degrees of freedom and the coordinates of the missile shoes.

There are no mass degrees of freedom of the unassembled structure at the missile shoes. It is evident that while the missile slides with respect to the rails, the mass matrix does not change nor, of course, does the K_z matrix change.

The stiffness matrix K, for a missile connected to and sliding along the rails, is now derived. The coordinates of the connected missile and rails are $\{q\}$. Since the coordinates of the unconnected missiles and rails are $\{z\}$, we can generate a matrix T which relates the $\{z\}$ and $\{q\}$ coordinates according to

$$\{z\} = T\{q\} \tag{1}$$

The T matrix is discussed in the next section. The coordinates $\{q\}$ do not include missile shoe coordinates. The stiffness matrix with respect to the coordinates $\{q\}$, for the assembled structure, is

$$K = K^t K_z T \tag{2}$$

It follows from Newton's law that

$$K\{q\} = \{P\} - M\{\ddot{q}\} \tag{3}$$

where $\{P\}$ is the applied forces matrix, variable with time, and M is the mass matrix, with both matrices being associated with the coordinates $\{q\}$. The equation is rewritten

$$\{\ddot{q}\} = M^{-1}\{\{P\} - K\{q\}\} \tag{4}$$

When the rails have initial curvature, the result on attaching the missile shoes to the rails would be internal forces in the structure, because the rails do not match the missile attach points, neglecting shoe clearances. Thus, rail initial curvature is a type of mismatch displacement. With rail initial curvature taken into account, the coordinates $\{q\}$ include this displacement. However, the initial curvature displacement does not cause forces from the rails, which are given by

$$K\{q\}$$

in Eq. (4). These forces must be subtracted, as shown in Eq. (5):

$$\{\ddot{q}\} = M^{-1}\{P\} - M^{-1}\left\{K\{q\} - \begin{bmatrix} 0 & 0 \\ 0 & K_2 \end{bmatrix}\begin{Bmatrix} 0 \\ \beta \end{Bmatrix}\right\} \tag{5}$$

This follows from forming the K_z matrix as

$$K_z = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \tag{6}$$

where K_1 relates to the missile and missile shoe coordinates and K_2 relates to the rail coordinates. For physical reasons, when the rail initial curvature displacement is given by the matrix $\{\beta\}$, the grid point accelerations are given by Eq. (5).

There are standard step-by-step numeric procedures for solving this system of equations. Note that for an inelastic missile, the stiffness matrix K is not time dependent and $K = K_2$.

It is sometimes desired to compute the missile rigid-body velocities just as the missile has left the rails. Matrix manipulations for this computation are in the Appendix.

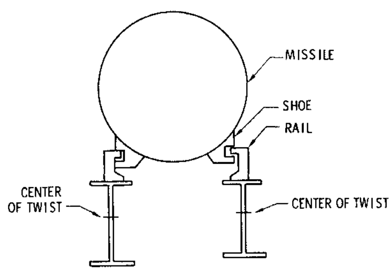
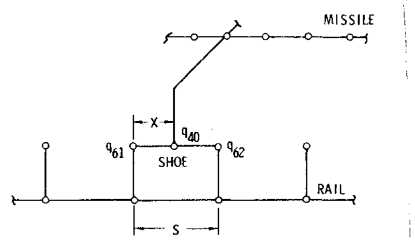


Fig. 3 Missile shoe/rail configuration.



X = DISTANCE BETWEEN RAIL COORDINATE q_{61} AND MISSILE SHOE COORDINATE q_{40}
S = DISTANCE BETWEEN RAIL COORDINATES q_{61} and q_{62}.

Fig. 4 Missile shoe and rail juncture.

IV. Coordinate Transformation

The transformation matrix T relates the coordinates of the unconnected missile and rail $\{z\}$ to the coordinates of the connected missile and rail $\{q\}$ according to Eq. (1).

The $\{z\}$ and $\{q\}$ coordinates can be numbered such that the T matrix is of the form

$$T = \begin{bmatrix} I & 0 \\ CM & CR \\ 0 & I \end{bmatrix} \tag{7}$$

The CM and CR matrices, which are functions of time, relate shoe coordinates to rail and missile coordinates. The number of rows in these matrices is the total number of missile shoe coordinates. The I matrices are unit matrices.

For each time interval, as the missile slides along the rails, the T and K matrices are recomputed to account for the new position of the missile relative to the rails. When a missile shoe is attached to a rail, the corresponding rows of the CM and CR matrices are formed on the basis of Eq. (8). Refer to Fig. 4, which shows an idealization of a typical missile support with the shoe attached to the rail. The coordinate of the missile shoe q_{40} is related to coordinates of the rails q_{61} and q_{62}, according to the lever law of mechanics, which gives

$$q_{40} = q_{62} + (q_{61} - q_{62})(s - x)/s \tag{8}$$

Thus, the necessary relationship involving coordinates is obtained for forming the T matrix. The physical significance of the relationship is that missile shoes are pinned to an infinitely stiff beam which, in turn, is pinned to two rail grid points. Computer logic determines which rail grid points are involved as the missile moves along the rails.

The T matrix which follows from this reasoning has the effect of lopping off part of the maximum bending moment for the static condition of a rail with a concentrated force, for example, as shown in Fig. 5, for a rail with built-in supports and eight rail grid points between supports. This approximation is an adequate representation of the true situation.

The missile shoe/rail configuration was shown in Fig. 3. It can be seen that due to design features and clearances, a missile shoe can become detached or partially detached from a rail. A fully attached missile shoe generates both a vertical and a transverse force, whereas a partially attached shoe generates a vertical or a transverse force. Of course, a missile shoe is completely detached from a rail when the shoe has slid past the rails. Finally, all the shoes are detached when the aft pair of shoes has slid past the rails.

If a missile shoe is not fully attached, the appropriate force in the equation

$$\{Z\} = K_z\{z\} \tag{9}$$

is zero. This equation furnishes the relationship between the missile shoe coordinates and other applicable coordinates. The formation of the appropriate rows of the CM and CR matrices for a detached or partially attached missile shoe is based on this relationship.

Computer logic can determine the missile shoe attachment condition. The forces $\{Z\}$ are grid point forces and among these are the missile shoe forces. Equation (9) can be used to determine the shoe attachment condition by computer logic, which determines whether or not the calculated shoe forces represent a physically possible vector.

V. Procedure of Application

A computer program based on the method described in the preceding paragraphs has been written. Prepared inputs which are independent of time are M, K_z, $\{\beta\}$, initial relative grid point location, shoe clearances, and missile mass. Time-dependent inputs are $\{P\}$ and missile thrust. Procedure of application is summarized as follows. 1) Compute $q_{t+\Delta t}$ from the following formula, which was derived in Ref. 6:

$$q_{t+\Delta t} = 2q_{t-\Delta t} - q_{t-3\Delta t} + \tfrac{4}{3}(\Delta t)^2(\ddot{q}_t + \ddot{q}_{t-\Delta t} + \ddot{q}_{t-2\Delta t})$$

2) Determine which rail grid points include each missile shoe. 3) Check missile shoes as to whether they are attached, partially attached, or detached from the rails. 4) Compute $\{P\}$. 5) Compute T to take into account the changed joint configuration. 6) Compute K as K depends on T. 7) Compute $\ddot{q}_{t+\Delta t}$ from Eq. (5). 8) Recompute $q_{t+\Delta t}$ from the Cowell formula, as derived in Ref. 6:

$$q_{t+\Delta t} = 2q_t - q_{t-\Delta t} + \tfrac{1}{12}(\Delta t)^2(\ddot{q}_{t+\Delta t} + 10\ddot{q}_t + \ddot{q}_{t-\Delta t})$$

9) Repeat step 3. 10) Recompute T and K if from step 9 a changed joint configuration is indicated. 11) Recompute $\ddot{q}_{t+\Delta t}$ from Eq. (5). 12) Go to the next time interval and repeat steps 1–11.

These steps are repeated automatically until the elapsed dynamic response time reaches an input value at which time the computer program terminates.

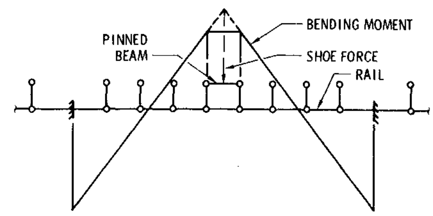


Fig. 5 Rail bending moment diagram.

VI. Numerical Experiment

The dynamic response of a missile and rail based on an idealization similar to that in Fig. 2 was computed. The missile had 25 mass degrees of freedom and the rail had 102 for a total of 127. There were three pairs of shoes, each shoe with two coordinates orthogonal to the missile longitudinal axis. The stiffness matrix of the missile, including shoe degrees-of-freedom, was 37 by 37. The stiffness matrices of the unconnected missile and rails and of the connected missile and rails were, respectively, 139 by 139, and 127 by 127.

The time interval for a stable step-by-step numerical procedure is proportional to the inverse of the structure frequency.[7] To keep this time interval as large as possible, it is desirable to eliminate the effects of higher frequency modes.

Therefore, the missile mass and stiffness matrices were formed by elimination of all but the lowest 25 modes from the original modes of an unsupported vehicle with 96 degrees-of-freedom. The rail mass matrix was adjusted by a procedure in which the elements of the inverse of the modal mass matrix corresponding to the 55 modes with the highest frequencies were set to zero.

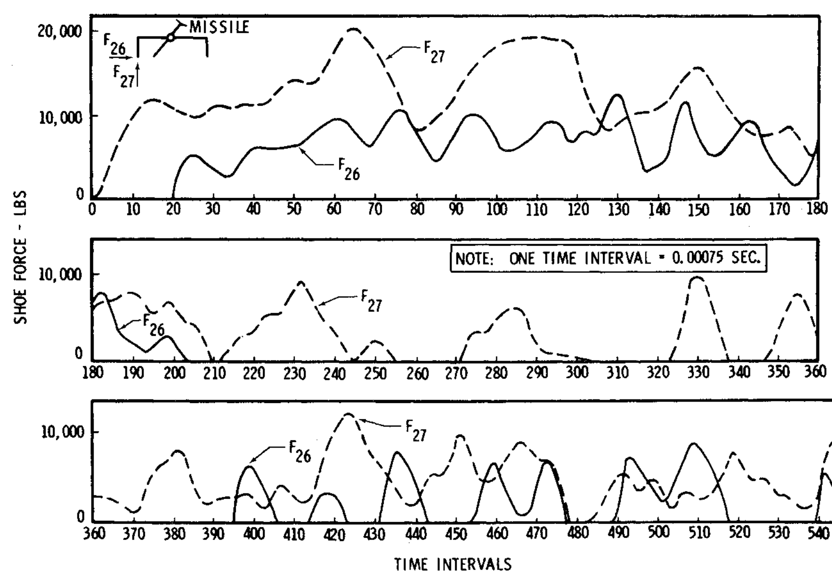


Fig. 6 Vertical and transverse shoe forces.

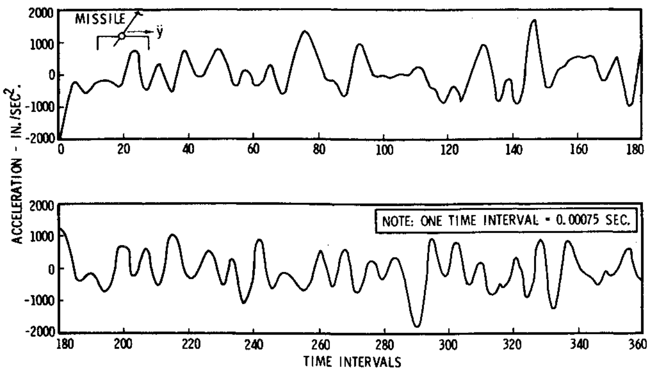


Fig. 7 Transverse missile station acceleration.

The numerical experiment was run for a total real time of 0.50 sec, or 666 time steps, with a step size of 0.00075 sec. The first pair of shoes slid off the rails at 0.3375 sec and the second at 0.4402 sec. At 0.50 sec, the aft pair of shoes was four time steps from being off the rails.

Excitation forces were caused by thrust misalignment and missile aerodynamic pressures. Initial shoe clearances in the vertical and transverse directions of the rails were 0.061 in. and 0.045 in., respectively, for the forward pair of shoes, and 0.094 in. and 0.065 in., respectively, for the middle and aft pairs of shoes. For this example, the rails were assumed to be initially straight.

Computer time on a Univac 1108 was 13 min to generate and store on tape the displacement, velocity, and acceleration of the coordinate representing the missile position along the rails and the 139 coordinates of the missile, missile shoes, and rails.

Some typical results are shown in Figs. 6–8, which describe the dynamic response of the missile station at the aft pair of shoes. Note that both vertical and transverse shoe forces (Fig. 6) are zero for significant time intervals. For this example, an assumption of constant contact of the shoes with the rails would be inaccurate. Although a missile station has both a vertical and a transverse acceleration, only the transverse acceleration is shown in Fig. 7. From examination of missile station displacements in Fig. 8, it is seen that initial shoe clearances are not small compared to missile station displacements, which indicates, for this example, that initial shoe clearances affect the dynamic response and especially the sequence of missile shoe attachment and detachment. This sequence showing the time history of detachment or attachment of each missile shoe force was also a computer output.

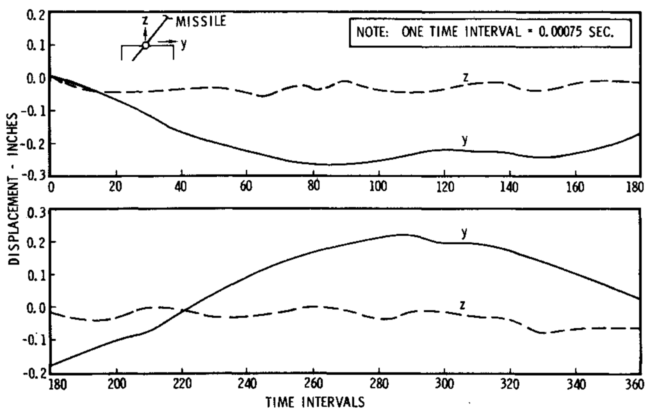


Fig. 8 Vertical and transverse missile station displacements.

Appendix: Missile Rigid-Body Velocities

If $\{F\}$ are external missile forces corresponding to the mass degrees-of-freedom or the missile coordinates $\{q_1\}$, the summation of the $\{F\}$ forces with respect to arbitrary reference axes is

$$\{P_R\} = D\{F\}$$

This equation defines the transformation matrix D, which is often generated in a static discrete-element computer program for checking purposes.

The momentum due to the velocities $\{\dot{q}_1\}$ is $M_1\{\dot{q}_1\}$ where M_1 is the missile mass matrix with respect to the coordinates $\{q_1\}$. The momentum with respect to the rigid body velocities $\{\dot{q}_R\}$ is $C\{\dot{q}_R\}$ where C is the rigid body mass matrix with respect to the coordinates $\{q_R\}$.

Momentum is a vector for which the technique for summing forces also holds. It follows that

$$C\{\dot{q}_R\} = DM_1\{\dot{q}_1\}$$

or

$$\{\dot{q}_R\} = C^{-1}DM_1\{\dot{q}_1\}$$

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